Broken S_3 Flavor Symmetry of Leptons and Quarks: Mass Spectra and Flavor Mixing Patterns

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Abstract

We apply the discrete S_3 flavor symmetry to both lepton and quark sectors of the standard model extended by introducing one Higgs triplet and realizing the type-II seesaw mechanism for finite neutrino masses. The resultant mass matrices of charged leptons (M_l), neutrinos (M_ν), up-type quarks ($M_{\rm u}$) and down-type quarks ($M_{\rm d}$) have a universal form consisting of two terms: one is proportional to the identity matrix I and the other is proportional to the democracy matrix D. We argue that the textures of M_l, $M_{\rm u}$ and $M_{\rm d}$ are dominated by the D term, while that of M_ν is dominated by the I term. This hypothesis implies a near mass degeneracy of three neutrinos and can naturally explain why the mass matrices of charged fermions are strongly hierarchical, why the quark mixing matrix is close to I and why the lepton mixing matrix contains two large angles. We discuss a rather simple perturbation ansatz to break the S_3 symmetry and obtain more realistic mass spectra of leptons and quarks as well as their flavor mixing patterns. We stress that the I term, which used to be ignored from M_l, $M_{\rm u}$ and $M_{\rm d}$, is actually important because it can significantly modify the smallest lepton flavor mixing angle θ_{13} or three quark flavor mixing angles.

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$\boxed{1}$ Flavor (or family) symmetries belong to one of a few promising approaches toward deeper understanding of the observed mass spectra and flavor mixing patterns of leptons and quarks [1, 2, 3]. Among a number of interesting discrete flavor symmetries discussed in the literature [4], the S_3 symmetry should be the simplest one and has been used to interpret the mass hierarchies of charged fermions [5] and predict the well-known "democratic" [6] and "tri-bimaximal" [7] neutrino mixing scenarios. A particular merit of the S_3 flavor symmetry is that it requires the mass matrices of three charged leptons (M_l), three neutrinos (M_ν), three up-type quarks ($M_{\rm u}$) and three down-type quarks ($M_{\rm d}$) to have a universal form $M_x = \xi_x I + \zeta_x D$ (for $x = l, \nu, {\rm u}, {\rm d}$), where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{1}$$

stand respectively for the identity and democracy matrices. Hence one may explore the flavor structures of leptons and quarks, both similarities and differences of their mass spectra and flavor mixing patterns, in a more or less parallel way.

The above point can be made clear if we extend the standard electroweak model by introducing one Higgs triplet and realizing the type-II seesaw mechanism [8] to generate finite neutrino masses. In this case the gauge-invariant Lagrangian relevant to quark and lepton masses reads

$$\begin{aligned} -\mathcal{L}_{\rm Q+L} &= \overline{Q_{\rm L}} Y_{\rm u} \tilde{H} U_{\rm R} + \overline{Q_{\rm L}} Y_{\rm d} H D_{\rm R} + \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \frac{1}{2} \overline{\ell_{\rm L}} Y_\Delta \Delta i\sigma_2 \ell^c_{\rm L} \\ & \quad - \lambda_\Delta M_\Delta H^T i \sigma_2 \Delta H + {\rm h.c.} \, , \end{aligned} \tag{2}$$

where $Q_{\rm L}$ and $\ell_{\rm L}$ are the left-handed doublets of quarks and leptons, $U_{\rm R}, D_{\rm R}, E_{\rm R}$ are the right-handed singlets of up-type quarks, down-type quarks and charged leptons, H and Δ stand respectively for the Higgs doublet and triplet. The last term of $\mathcal{L}_{\rm Q+L}$ is lepton-number-violating, and thus its dimensionless coefficient λ_Δ is naturally small [9]. Once the neutral components of H and Δ acquire their vacuum expectation values, the $SU(2)_{\rm L} \times U(1)_{\rm Y}$ gauge symmetry is spontaneously broken such that all of the quarks and leptons become massive:

$$-\mathcal{L}_{\rm mass} = \overline{U_{\rm L}} M_{\rm u} U_{\rm R} + \overline{D_{\rm L}} M_{\rm d} D_{\rm R} + \overline{E_{\rm L}} M_l E_{\rm R} + \frac{1}{2} \overline{\nu_{\rm L}} M_\nu \nu^c_{\rm L} + {\rm h.c.} \, , \tag{3}$$

where the charged-fermion mass matrices are given by $M_x \equiv Y_x \langle H \rangle$ (for $x = {\rm u}, {\rm d}, l$) with $\langle H \rangle \approx 174$ GeV, and the Majorana neutrino mass matrix is given by $M_\nu = Y_\Delta \langle \Delta \rangle$ with $\langle \Delta \rangle = 2\lambda_\Delta \langle H \rangle^2 / M_\Delta$. The mass scale of the Higgs triplet M_Δ is expected to be much higher than the electroweak scale characterized by $\langle H \rangle$, so the tiny mass scale of M_ν is apparently attributed to the smallness of λ_Δ and the largeness of M_Δ. Current experimental data on the electroweak ρ parameter require $\langle \Delta \rangle < 2$ GeV [5]. Now we apply the non-Abelian discrete symmetry S_3, a permutation group of three objects, to the lepton and quark sectors. It contains six elements,

$$S^{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad S^{(213)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

$$
\begin{aligned}
S^{(132)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad S^{(321)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \\
S^{(312)} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \quad S^{(231)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ,
\end{aligned} \tag{4}
$$

which fall into three conjugacy classes. Allowing the lepton and quark fields to transform as $Q_{\rm L} \to S^{(ijk)} Q_{\rm L}$, $\ell_{\rm L} \to S^{(ijk)} \ell_{\rm L}$, $U_{\rm R} \to S^{(ijk)} U_{\rm R}$, $D_{\rm R} \to S^{(ijk)} D_{\rm R}$ and $E_{\rm R} \to S^{(ijk)} E_{\rm R}$, one may easily find that the gauge interactions (or kinetic terms) of leptons and quarks are invariant. The Yukawa interactions of leptons and quarks in Eq. (2) are also invariant under the same transformations if the following commutation conditions are satisfied:

$$
\left[Y_{\rm u} \, , S^{(ijk)} \right] = \left[Y_{\rm d} \, , S^{(ijk)} \right] = \left[Y_{l} \, , S^{(ijk)} \right] = \left[Y_{\Delta} \, , S^{(ijk)} \right] = \mathbf{0} \; . \tag{5}
$$

The mass matrices M_x (for $x = {\rm u}, {\rm d}, l, \nu$) must equivalently satisfy $\left[M_x \, , S^{(ijk)} \right] = \mathbf{0}$, implying that they can only take a universal form $M_x = \xi_x I + \zeta_x D$ in the S_3 symmetry limit with $I = S^{(123)}$ and $D = S^{(213)} + S^{(132)} + S^{(321)} = S^{(123)} + S^{(312)} + S^{(231)}$ [10]. Because an orthogonal matrix with two large rotation angles is needed to diagonalize the democracy matrix D, such as

$$
\begin{aligned}
U_0^T D U_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv H_2 \, , \quad U_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & -\sqrt{3} \end{pmatrix} ; \\
V_0^T D V_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \equiv H_3 \, , \quad V_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix} ,
\end{aligned} \tag{6}
$$

it is potentially possible to explain the large solar and atmospheric neutrino mixing angles. One can see that V_0^T corresponds to the democratic mixing pattern [6] while U_0 is just the tri-bimaximal mixing pattern [7]. Since H_3 contains a non-vanishing (dominant) matrix element in the (3,3) position, it is consistent with the observed mass hierarchies of charged fermions (i.e., $m_t \gg m_c \gg m_u$, $m_b \gg m_s \gg m_d$ and $m_\tau \gg m_\mu \gg m_e$) in the symmetry limit. This observation means that $\xi_x \ll \zeta_x$ (for $x = {\rm u}, {\rm d}, l$) is likely to hold for charged fermions, while $\xi_\nu \gg \zeta_\nu$ is more reasonable to generate two large lepton flavor mixing angles together with a nearly degenerate neutrino mass spectrum [6, 11, 12, 13, 14]. Of course, proper perturbations to the lepton or quark mass matrices are necessary [3, 15] in order to stabilize the dominant term of the quark or lepton flavor mixing matrix and produce appreciable CP violation in the lepton or quark sector.

In this letter we stress that the I term of M_x, which used to be ignored for charged fermions (i.e., $\xi_l = \xi_{\rm u} = \xi_{\rm d} = 0$ was often assumed or dictated by the $S_{3{\rm L}} \times S_{3{\rm R}}$ symmetry), is actually important. On the one hand, this term is present in the S_3 symmetry limit and thus has no good reason to be switched off. On the other hand, small ξ_l can significantly modify the smallest lepton flavor mixing angle θ_{13}, while small $\xi_{\rm u}$ and $\xi_{\rm d}$ can remarkably modify three quark flavor mixing angles. We illustrate our points by discussing a rather simple perturbation ansatz to explicitly break the S_3 symmetry and obtain more realistic mass spectra of leptons and quarks as well as their flavor mixing patterns.

$\boxed{\mathbf{2}}$ Let us first consider the lepton sector. Based on the S_3 symmetry discussed above, the mass matrices of charged leptons and neutrinos can be written as

$$
\begin{aligned}
M_l &= \frac{c_l}{3}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + r_l \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] + \Delta M_l \; , \\
M_\nu &= c_\nu \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_\nu \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right] + \Delta M_\nu \; ,
\end{aligned}
\tag{7}
$$

where $c_x > 0$, r_x is real and $|r_x| \ll 1$ holds (for $x = l$ and ν), and the explicit symmetry-breaking terms ΔM_l and ΔM_ν are assumed to take the diagonal forms [6, 13]

$$
\Delta M_l = \frac{c_l}{3}\begin{pmatrix} -i\delta_l & 0 & 0 \\ 0 & +i\delta_l & 0 \\ 0 & 0 & \varepsilon_l \end{pmatrix} \; , \quad \Delta M_\nu = c_\nu \begin{pmatrix} -\delta_\nu & 0 & 0 \\ 0 & +\delta_\nu & 0 \\ 0 & 0 & \varepsilon_\nu \end{pmatrix}
\tag{8}
$$

with $0 < \delta_x \ll \varepsilon_x \ll 1$ (for $x = l$ and ν). To diagonalize M_l, one may transform it into the following hierarchical texture:

$$
M'_l \equiv V_0^T M_l V_0 = \frac{c_l}{9\sqrt{3}} \begin{pmatrix} 3\sqrt{3} r_l & -i3\delta_l & -i3\sqrt{2}\delta_l \\ -i3\delta_l & \sqrt{3}\left(3r_l + 2\varepsilon_l\right) & -\sqrt{6}\varepsilon_l \\ -i3\sqrt{2}\delta_l & -\sqrt{6}\varepsilon_l & \sqrt{3}\left(9 + 3r_l + \varepsilon_l\right) \end{pmatrix} \; .
\tag{9}
$$

where V_0 has been given in Eq. (6). In the assumption of $|r_l| \ll \varepsilon_l$, we can diagonalize M'_l and obtain the masses of three charged leptons to a good degree of accuracy:

$$
m_\tau \approx c_l \left(1 + \frac{\varepsilon_l}{9} + \frac{r_l}{3}\right) \; , \quad m_\mu \approx c_l \left(\frac{2\varepsilon_l}{9} + \frac{r_l}{3}\right) \; , \quad m_e \approx c_l \left|\frac{\delta_l^2}{6\varepsilon_l} + \frac{r_l}{3}\right| \; .
\tag{10}
$$

Defining $m_0 \equiv c_l r_l/3$, we have $|m_0| < m_\mu$. Then ε_l and δ_l are given by

$$
\varepsilon_l \approx \frac{9}{2}\frac{m_\mu - m_0}{m_\tau - m_0} \; , \quad \delta_l \approx \frac{2\varepsilon_l}{\sqrt{3}} \frac{\sqrt{|m_e - |m_0||}}{\sqrt{m_\mu - m_0}} \; .
\tag{11}
$$

As a consequence of Eq. (10), the constraint $m_0 < m_e$ should be satisfied for $m_0 > 0$. The unitary matrix used to diagonalize M_l (i.e., $V_l^\dagger M_l V_l^* = \mathrm{Diag}\{m_e, m_\mu, m_\tau\}$) is found to deviate from V_0 at the level of $\mathcal{O}(\varepsilon_l)$ and $\mathcal{O}(\delta_l)$:

$$
V_l \approx V_0 - \frac{i}{\sqrt{6}} \frac{\sqrt{|m_e - |m_0||}}{\sqrt{m_\mu - m_0}} \begin{pmatrix} 1 & \sqrt{3} & 0 \\ 1 & -\sqrt{3} & 0 \\ -2 & 0 & 0 \end{pmatrix} + \frac{1}{2\sqrt{3}} \frac{m_\mu - m_0}{m_\tau - m_0} \begin{pmatrix} 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix} \; .
\tag{12}
$$

On the other hand, the unitary matrix used to diagonalize the Majorana neutrino mass matrix M_ν (i.e., $V_\nu^\dagger M_\nu V_\nu^* = \mathrm{Diag}\{m_1, m_2, m_3\}$) is approximately given by [13]

$$
V_\nu \approx \frac{1}{\varepsilon_\nu} \begin{pmatrix} \varepsilon_\nu c_\theta & \varepsilon_\nu s_\theta & r_\nu \\ -\varepsilon_\nu s_\theta & \varepsilon_\nu c_\theta & r_\nu \\ r_\nu \left(s_\theta - c_\theta\right) & -r_\nu \left(s_\theta + c_\theta\right) & \varepsilon_\nu \end{pmatrix} \; ,
\tag{13}
$$

where $c_\theta \equiv \cos\theta$ and $s_\theta \equiv \sin\theta$ with $\tan 2\theta \equiv r_\nu/\delta_\nu$. Three neutrino mass eigenvalues are

$$
\begin{aligned}
m_3 &\approx c_\nu \left(1 + r_\nu + \varepsilon_\nu\right) \, , \\
m_2 &\approx c_\nu \left(1 + r_\nu + \sqrt{r_\nu^2 + \delta_\nu^2}\right) \, , \\
m_1 &\approx c_\nu \left(1 + r_\nu - \sqrt{r_\nu^2 + \delta_\nu^2}\right) \, .
\end{aligned} \tag{14}
$$

Therefore, $\Delta m^2_{21} \approx 4c_\nu^2\sqrt{r_\nu^2 + \delta_\nu^2}$ and $\Delta m^2_{32} \approx 2c_\nu^2\varepsilon_\nu$, from which $r_\nu/\varepsilon_\nu \approx c_\theta s_\theta \Delta m^2_{21}/\Delta m^2_{32}$ can be obtained. The 3×3 lepton flavor mixing matrix is defined as $V_{\rm MNS} = V_l^\dagger V_\nu$ [16]. A straightforward calculation leads us to

$$
\begin{aligned}
V_{\rm MNS} \approx\; & \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3}\,(c_\theta + s_\theta) & -\sqrt{3}\,(c_\theta - s_\theta) & 0 \\ (c_\theta - s_\theta) & (c_\theta + s_\theta) & -2 \\ \sqrt{2}\,(c_\theta - s_\theta) & \sqrt{2}\,(c_\theta + s_\theta) & \sqrt{2} \end{pmatrix} \\
& + \frac{i}{\sqrt{6}} \frac{\sqrt{|m_e - |m_0||}}{\sqrt{m_\mu - m_0}} \begin{pmatrix} (c_\theta - s_\theta) & (c_\theta + s_\theta) & -2 \\ \sqrt{3}\,(c_\theta + s_\theta) & -\sqrt{3}\,(c_\theta - s_\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
& + \frac{1}{2\sqrt{3}} \frac{m_\mu - m_0}{m_\tau - m_0} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\,(c_\theta - s_\theta) & \sqrt{2}\,(c_\theta + s_\theta) & \sqrt{2} \\ -\left(c_\theta - s_\theta\right) & -\left(c_\theta + s_\theta\right) & 2 \end{pmatrix} \\
& + \frac{1}{\sqrt{6}} \frac{r_\nu}{\varepsilon_\nu} \begin{pmatrix} 0 & 0 & 0 \\ 2\,(c_\theta - s_\theta) & 2\,(c_\theta + s_\theta) & 2 \\ -\sqrt{2}\,(c_\theta - s_\theta) & -\sqrt{2}\,(c_\theta + s_\theta) & 2\sqrt{2} \end{pmatrix} \, .
\end{aligned} \tag{15}
$$

Comparing this result with the standard parametrization of $V_{\rm MNS}$ advocated in Ref. [5] (see also Ref. [17]), we immediately arrive at three flavor mixing angles

$$
\begin{aligned}
\theta_{12} &\approx \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{r_\nu}{\delta_\nu}\right) \, , \\
\theta_{13} &\approx \arcsin\left(\frac{2}{\sqrt{6}} \frac{\sqrt{|m_e - |m_0||}}{\sqrt{m_\mu - m_0}}\right) \, , \\
\theta_{23} &\approx \frac{1}{2}\arcsin\left[\frac{2\sqrt{2}}{3}\left(1 + \frac{1}{2}\frac{m_\mu - m_0}{m_\tau - m_0} + \frac{r_\nu}{\varepsilon_\nu}\right)\right] \, ,
\end{aligned} \tag{16}
$$

together with the Dirac CP-violating phase $\delta \approx \pi/2$. Two Majorana CP-violating phases are trivial in this scenario, and thus the effective mass of the neutrinoless double-beta decay is simply given by $\langle m\rangle_{ee} \approx c_\nu$. Different from Ref. [13], here both θ_{13} and θ_{23} get modified because of the non-vanishing $m_0 = c_l r_l/3$ which signifies an important contribution from the I term in the S_3 symmetry limit. Some discussions are in order.

- A global analysis of current neutrino oscillation data yields $\theta_{12} \approx 34.5^\circ$ [18], implying $\theta \approx 10.5^\circ$ or equivalently $r_\nu/\delta_\nu \approx 0.38$. We see that the size of θ is smaller than the Cabibbo angle of quark flavor mixing (i.e., $\theta_{\rm C} \approx 13^\circ$ [5]), so the so-called quark-lepton complementarity relation $\theta_{12} + \theta_{\rm C} \approx \pi/4$ becomes less favored than before. In addition, we obtain $r_\nu/\varepsilon_\nu \approx 5.7\times 10^{-3}$ from $r_\nu/\varepsilon_\nu \approx c_\theta s_\theta \Delta m^2_{21}/\Delta m^2_{32}$ with the typical inputs $\Delta m^2_{21} \approx 7.6\times 10^{-5}$ eV^2 and $\Delta m^2_{32} \approx 2.4\times 10^{-3}$ eV^2 [18].

- If $|m_0|$ is significantly larger than m_e, the smallest neutrino mixing angle θ_{13} will be increased. In this case, the magnitude of θ_{13} is sensitive to that of m_0 , and thus the latter can get constrained from the present experimental data. With the help of Eq. (16), we obtain

$$m_0 \approx -\frac{2m_e + 3m_\mu \sin^2\theta_{13}}{2 - 3\sin^2\theta_{13}} \approx -\left(m_e + \frac{3}{2}m_\mu \sin^2\theta_{13}\right) , \qquad (17)$$

for $m_0 < 0$ and $|m_0| > m_e$. Given $m_e \approx 0.4866$ MeV and $m_\mu \approx 102.718$ MeV at the electroweak scale [19], the upper bound $\theta_{13} \leq 12^\circ$ [18] leads us to $-14.7m_e \leq m_0 < -m_e$ at the same energy scale. In view of $r_l = 3m_0/c_l \approx 3m_0/m_\tau$ together with $m_\tau \approx 1746.24$ MeV at the electroweak scale [19], we find $-1.2 \times 10^{-2} \leq r_l < -8.4 \times 10^{-4}$. So the magnitude of r_l is strongly suppressed. Of course, we have $\theta_{13} \approx 0^\circ$ for $|m_0| \approx m_e$ and $\theta_{13} \approx \arcsin\sqrt{2m_e/(3m_\mu)} \approx 3.2^\circ$ for $m_0 \approx 0$. Once the value of θ_{13} is experimentally fixed, one may then be able to determine the important S_3-symmetry parameter m_0.

- Because of $|m_0|/m_\mu < 15m_e/m_\mu \approx 0.071$ at the electroweak scale, the m_0-induced correction to θ_{23} in Eq. (16) is apparently negligible. Thus $\varepsilon_l \approx 9m_\mu/(2m_\tau) \approx 0.26$ and $\delta_l \approx \sqrt{2}\varepsilon_l \sin\theta_{13} \leq 7.6\times10^{-2}$. The smallness of r_ν/ε_ν also makes its contribution to θ_{23} negligible. Taking $m_\mu/m_\tau \approx 0.0588$ [19], we obtain $\theta_{23} \approx 38^\circ$. This result, which certainly depends on the assumed perturbation forms of ΔM_l and ΔM_ν, is a bit lower than the best-fit value of θ_{23} (i.e., $\theta_{23} = 42.8^\circ$ [18]) but lies in the 3σ interval of θ_{23} (i.e., $\theta_{23} = 42.8^{+10.7^\circ}_{-7.3^\circ}$ [18]).

Note that the magnitude of r_ν can be determined only after the absolute mass scale of three neutrinos (i.e., c_ν) is known. Taking $c_\nu \approx 0.1$ eV for example, we find $\varepsilon_\nu \approx \Delta m^2_{32}/(2c^2_\nu) \approx 0.12$ and therefore $r_\nu \approx 6.8 \times 10^{-4}$. The smallness of both r_l and r_ν is consistent with our original expectations. Although r_l and r_ν are two free parameters, they are intrinsical in the S_3 flavor symmetry and hence should be taken into account. We have demonstrated that both of them are phenomenologically important in understanding the lepton mass hierarchy and the flavor mixing pattern.

$\boxed{\mathbf{3}}$ We proceed to look at the quark sector. Based on the S_3 flavor symmetry, the mass matrices of up- and down-type quarks are written as

$$\begin{aligned} M_{\rm u} &= \frac{c_{\rm u}}{3}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + r_{\rm u}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] + \Delta M_{\rm u} , \\ M_{\rm d} &= \frac{c_{\rm d}}{3}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + r_{\rm d}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] + \Delta M_{\rm d} , \end{aligned} \qquad (18)$$

where $c_x > 0$, r_x is real and $|r_x| \ll 1$ holds (for $x =$ u and d). For the sake of simplicity, the textures of $\Delta M_{\rm u}$ and $\Delta M_{\rm d}$ are taken to be exactly parallel to those of ΔM_ν and ΔM_l given in Eq. (8). In other words,

$$\Delta M_{\rm u} = \frac{c_{\rm u}}{3}\begin{pmatrix} -\delta_{\rm u} & 0 & 0 \\ 0 & +\delta_{\rm u} & 0 \\ 0 & 0 & \varepsilon_{\rm u} \end{pmatrix} , \quad \Delta M_{\rm d} = \frac{c_{\rm d}}{3}\begin{pmatrix} -i\delta_{\rm d} & 0 & 0 \\ 0 & +i\delta_{\rm d} & 0 \\ 0 & 0 & \varepsilon_{\rm d} \end{pmatrix} \qquad (19)$$

with $0 < \delta_x \ll \varepsilon_x \ll 1$ (for $x = \mathrm{u}$ and d). The diagonalization of M_u and M_d is almost the same as that of M_l. As a result,

$$\begin{aligned}
m_t &= c_\mathrm{u}\left(1+\frac{\varepsilon_\mathrm{u}}{9}+\frac{r_\mathrm{u}}{3}\right) , \quad m_c = c_\mathrm{u}\left(\frac{2\varepsilon_\mathrm{u}}{9}+\frac{r_\mathrm{u}}{3}\right) , \quad m_u = c_\mathrm{u}\left|\frac{r_\mathrm{u}}{3}-\frac{\delta_\mathrm{u}^2}{6\varepsilon_\mathrm{u}}\right| ; \\
m_b &= c_\mathrm{d}\left(1+\frac{\varepsilon_\mathrm{d}}{9}+\frac{r_\mathrm{d}}{3}\right) , \quad m_s = c_\mathrm{d}\left(\frac{2\varepsilon_\mathrm{d}}{9}+\frac{r_\mathrm{d}}{3}\right) , \quad m_d = c_\mathrm{d}\left|\frac{r_\mathrm{d}}{3}+\frac{\delta_\mathrm{d}^2}{6\varepsilon_\mathrm{d}}\right| . \qquad (20)
\end{aligned}$$

Defining $m_0' = c_\mathrm{u} r_\mathrm{u}/3$ and $m_0'' = c_\mathrm{d} r_\mathrm{d}/3$, we have

$$\begin{aligned}
\varepsilon_\mathrm{u} &= \frac{9}{2}\frac{m_c - m_0'}{m_t - m_0'} , \quad \delta_\mathrm{u} \approx \frac{2\varepsilon_\mathrm{u}}{\sqrt{3}}\frac{\sqrt{|m_u - |m_0'||}}{\sqrt{m_c - m_0'}} , \\
\varepsilon_\mathrm{d} &= \frac{9}{2}\frac{m_s - m_0''}{m_b - m_0''} , \quad \delta_\mathrm{d} \approx \frac{2\varepsilon_\mathrm{d}}{\sqrt{3}}\frac{\sqrt{|m_d - |m_0''||}}{\sqrt{m_s - m_0''}} , \qquad (21)
\end{aligned}$$

where the phenomenologically uninteresting case with $0 < m_0' < m_u$ has been discarded. Furthermore, Eq. (20) gives rise to the constraints $|m_0'| < m_u$ for $m_0' < 0$ and $m_0'' < m_d$ for $m_0'' > 0$. Given $V_\mathrm{u}^\dagger M_\mathrm{u} V_\mathrm{u}^* = \mathrm{Diag}\{m_u, m_c, m_t\}$ and $V_\mathrm{d}^\dagger M_\mathrm{d} V_\mathrm{d}^* = \mathrm{Diag}\{m_d, m_s, m_b\}$, a careful perturbation calculation yields

$$\begin{aligned}
V_\mathrm{u} \approx\; & V_0 + \frac{x_\mathrm{u}}{\sqrt{6}}\begin{pmatrix} 1 & -\sqrt{3} & 0 \\ 1 & \sqrt{3} & 0 \\ -2 & 0 & 0 \end{pmatrix} + \frac{y_\mathrm{u}}{2\sqrt{3}}\begin{pmatrix} 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix} \\
& - \frac{x_\mathrm{u}^2}{2\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix} + \frac{x_\mathrm{u} y_\mathrm{u}}{\sqrt{6}}\begin{pmatrix} 3 & 0 & -\sqrt{6} \\ 3 & 0 & \sqrt{6} \\ 3 & 0 & 0 \end{pmatrix} , \\
V_\mathrm{d} \approx\; & V_0 - \frac{i x_\mathrm{d}}{\sqrt{6}}\begin{pmatrix} 1 & \sqrt{3} & 0 \\ 1 & -\sqrt{3} & 0 \\ -2 & 0 & 0 \end{pmatrix} + \frac{y_\mathrm{d}}{2\sqrt{3}}\begin{pmatrix} 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix} \\
& - \frac{x_\mathrm{d}^2}{2\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix} - \frac{i x_\mathrm{d} y_\mathrm{d}}{\sqrt{6}}\begin{pmatrix} 3 & 0 & \sqrt{6} \\ 3 & 0 & -\sqrt{6} \\ 3 & 0 & 0 \end{pmatrix} , \qquad (22)
\end{aligned}$$

where $x_\mathrm{q} \equiv \sqrt{3}\delta_\mathrm{q}/(2\varepsilon_\mathrm{q})$ and $y_\mathrm{q} \equiv 2\varepsilon_\mathrm{q}/9$ (for $\mathrm{q} = \mathrm{u}$ or d) have been defined. The quark flavor mixing matrix $V_\mathrm{CKM} \equiv V_\mathrm{u}^\dagger V_\mathrm{d}$ [20] turns out to be

$$V_\mathrm{CKM} \approx \begin{pmatrix} 1 & x_\mathrm{u} - i x_\mathrm{d} & -\dfrac{y_\mathrm{d}\,(2 i x_\mathrm{d} + x_\mathrm{u}) - 3 x_\mathrm{u} y_\mathrm{u}}{\sqrt{2}} \\ -x_\mathrm{u} - i x_\mathrm{d} & 1 & \dfrac{y_\mathrm{u} - y_\mathrm{d}}{\sqrt{2}} \\ \dfrac{i x_\mathrm{d}(y_\mathrm{u} - 3 y_\mathrm{d}) - 2 x_\mathrm{u} y_\mathrm{u}}{\sqrt{2}} & -\dfrac{y_\mathrm{u} - y_\mathrm{d}}{\sqrt{2}} & 1 \end{pmatrix} , \qquad (23)$$

where the $\mathcal{O}(x_\mathrm{u}^2)$, $\mathcal{O}(x_\mathrm{d}^2)$, $\mathcal{O}(x_\mathrm{u} x_\mathrm{d})$ and $\mathcal{O}(y_\mathrm{u} y_\mathrm{d})$ corrections to the diagonal elements of V_CKM have been neglected. This result clearly shows that the dominant term of V_CKM is the identity matrix I, and thus its three flavor mixing angles must be small and sensitive

to the values of m_0' and m_0''. The latter should not be ignored not only because they have an impact on $V_{\rm CKM}$ but also because they come from an intrinsic S_3 symmetry term.

To examine whether Eq. (23) is compatible with current experimental data or not, we make use of the following values of quark masses at the electroweak scale [19]:

$$
\begin{aligned}
m_u &= (0.85 \sim 1.77)\ \text{MeV}\ , & m_d &= (1.71 \sim 4.14)\ \text{MeV}\ , \\
m_c &= (0.535 \sim 0.703)\ \text{GeV}\ , & m_s &= (40 \sim 71)\ \text{MeV}\ , \\
m_t &= (168.7 \sim 174.7)\ \text{GeV}\ , & m_b &= (2.80 \sim 2.98)\ \text{GeV}\ .
\end{aligned}
\tag{24}
$$

One may roughly take the geometric relations $m_u/m_c \approx m_c/m_t \approx 1/300$ and $m_d/m_s \approx m_s/m_b \approx 1/40$ to illustrate the strong quark mass hierarchies. In our discussions $m_0' \ll m_c$ and $m_0'' \ll m_s$ are reasonably assumed to assure that the perturbation calculations are valid. So the magnitudes of $\delta_{\rm u}$ and $\delta_{\rm d}$ are sensitive to m_0' and m_0'', whereas those of $\varepsilon_{\rm u}$ and $\varepsilon_{\rm d}$ are not. In other words, $x_{\rm u}$ (or $x_{\rm d}$) may significantly deviate from $\sqrt{m_u/m_c}$ (or $\sqrt{m_d/m_s}$) while $y_{\rm u}$ (or $y_{\rm d}$) is essentially equal to m_c/m_t (or m_s/m_b). These observations are helpful when we confront Eq. (23) with the experimental constraints on $V_{\rm CKM}$ [5],

$$
\begin{aligned}
|V_{\rm CKM}| &= \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \\
&= \begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.0010 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.0010 & 0.97334 \pm 0.00023 & 0.0415^{+0.0010}_{-0.0011} \\ 0.00874^{+0.00026}_{-0.00037} & 0.0407 \pm 0.0010 & 0.999133^{+0.000044}_{-0.000043} \end{pmatrix} .
\end{aligned}
\tag{25}
$$

Some discussions are in order.

- We obtain $|V_{cb}| \approx |V_{ts}| \approx |y_{\rm u} - y_{\rm d}|/\sqrt{2} \approx (m_s/m_b - m_c/m_t)/\sqrt{2}$ in the leading-order approximation, which is too low to fit the observed values of $|V_{cb}|$ and $|V_{ts}|$. The reason for this discrepancy is simple: the diagonal patterns of $\Delta M_{\rm u}$ and $\Delta M_{\rm d}$ taken in Eq. (19) fail in producing a sufficiently large mixing angle between the second and third families in the quark sector. In fact, a similar problem has appeared in the lepton sector: the diagonal perturbation matrices ΔM_l and ΔM_ν in Eq. (8) are unable to produce $\theta_{23} \approx \pi/4$ for the atmospheric neutrino mixing angle. A straightforward way out should be to find out a class of different patterns of ΔM_x (for $x = l, \nu, {\rm u}, {\rm d}$), which might be off-diagonal or partially off-diagonal but can simultaneously make the mixing angle θ_{23} of $V_{\rm MNS}$ and the matrix elements $|V_{cb}|$ and $|V_{ts}|$ of $V_{\rm CKM}$ large enough [21].

- Here we are concerned about whether it is possible to achieve the experimentally-favored values of $|V_{ub}|/|V_{cb}|$ and $|V_{td}|/|V_{ts}|$ in the presence of non-vanishing m_0' and m_0'', because many of the hitherto-proposed quark mass ansätze predict $|V_{ub}|/|V_{cb}| \approx \sqrt{m_u/m_c}$ [22] which is badly lower than the present experimental value of $|V_{ub}|/|V_{cb}|$. Eq. (23) leads us to

$$
\frac{|V_{ub}|}{|V_{cb}|} = \frac{\sqrt{4x_{\rm d}^2 y_{\rm d}^2 + x_{\rm u}^2 (y_{\rm d} - 3y_{\rm u})^2}}{|y_{\rm u} - y_{\rm d}|}\ , \quad \frac{|V_{td}|}{|V_{ts}|} = \frac{\sqrt{4x_{\rm u}^2 y_{\rm u}^2 + x_{\rm d}^2 (y_{\rm u} - 3y_{\rm d})^2}}{|y_{\rm u} - y_{\rm d}|}\ .
\tag{26}
$$

In the $m_0' = m_0'' \approx 0$ case, we simply obtain $|V_{ub}|/|V_{cb}| \approx 2\sqrt{m_d/m_s}$ and $|V_{td}|/|V_{ts}| \approx 3\sqrt{m_d/m_s}$ from Eq. (26). These two results are apparently in conflict with the experimental data given in Eq. (25). Hence one has to switch on the contributions from m_0' and m_0''. To see how large m_0' and m_0'' should be, we rewrite Eq. (26) as

$$
\begin{aligned}
x_{\rm u}^2 &= \frac{1}{3\left(3R^2 - 14R + 3\right)} \left[(R-3)^2 \frac{|V_{ub}|^2}{|V_{cb}|^2} - 4\frac{|V_{td}|^2}{|V_{ts}|^2} \right] \, , \\
x_{\rm d}^2 &= \frac{1}{3\left(3R^2 - 14R + 3\right)} \left[(1-3R)^2 \frac{|V_{td}|^2}{|V_{ts}|^2} - 4R^2\frac{|V_{ub}|^2}{|V_{cb}|^2} \right] \, ,
\end{aligned}
\tag{27}
$$

where $R \equiv y_{\rm u}/y_{\rm d}$. Because $y_{\rm u}$ and $y_{\rm d}$ are almost insensitive to m_0' and m_0'', we approximately have $R \approx (m_c m_b)/(m_t m_s) \in [0.12,\ 0.31]$ from Eq. (24). Typically taking $R \approx 0.3$ and adopting the best-fit values of $|V_{ub}|/|V_{cb}|$ and $|V_{td}|/|V_{ts}|$ given in Eq. (25), we find $x_{\rm u}^2 \approx 4.6 \times 10^{-2}$ and $x_{\rm d}^2 \approx 8.0 \times 10^{-4}$. Accordingly, we get $||m_0'| - m_u| \approx m_c x_{\rm u}^2 \approx 14 m_u$ and $||m_0''| - m_d| \approx m_s x_{\rm d}^2 \approx 3.2 \times 10^{-2} m_d$ if we simply input $m_u/m_c \approx 1/300$ and $m_d/m_s \approx 1/40$. In view of $r_{\rm u} = 3m_0'/c_{\rm u} \approx 3m_0'/m_t$ and $r_{\rm d} = 3m_0''/c_{\rm d} \approx 3m_0''/m_b$, we obtain $r_{\rm u} \approx 45 m_u/m_t \approx 5 \times 10^{-4}$ (for $m_0' > m_u$) as well as $r_{\rm d} \approx \pm 3 m_d/m_b \approx \pm 1.9 \times 10^{-3}$ (for $|m_0''| \sim m_d$). The smallness of both $r_{\rm u}$ and $r_{\rm d}$ is consistent with the strong mass hierarchies of up- and down-type quarks which have been described by the democratic D term in the S_3 flavor symmetry limit.

- In this ansatz the Cabibbo angle $\theta_{\rm C}$ (i.e., $\sin\theta_{\rm C} \equiv |V_{us}| \approx |V_{cd}|$) is given by

$$
\theta_{\rm C} \; = \; \arcsin\left(|V_{us}|\right) \; \approx \; \arcsin\left(|V_{cd}|\right) \; \approx \; \arcsin\left(\sqrt{x_{\rm u}^2 + x_{\rm d}^2}\right) \, . \tag{28}
$$

With the inputs $x_{\rm u}^2 \approx 4.6 \times 10^{-2}$ and $x_{\rm d}^2 \approx 8.0 \times 10^{-4}$ obtained above, we obtain $\theta_{\rm C} \approx 12.5^\circ$ or equivalently $|V_{us}| \approx |V_{cd}| \approx 0.22$. Such a result is quantitatively compatible with current experimental data, but it is qualitatively different from the conventional prediction $\sin\theta_{\rm C} \approx \sqrt{m_d/m_s}$ or $\sin\theta_{\rm C} \approx |\sqrt{m_d/m_s} - e^{i\phi}\sqrt{m_u/m_c}|$ with ϕ being a free phase parameter based on a class of quark mass ansätze [23, 24, 25].

- Since CP violation has been observed in the quark sector, let us take a look at the consequences of our ansatz on four well-known rephasing invariants of CP violation: the Jarlskog parameter $\mathcal{J}$ [26] and three inner angles of the CKM unitarity triangle $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ [5]. In the leading-order approximation, we obtain

$$
\mathcal{J} \; \equiv \; {\rm Im}\left(V_{us}V_{cb}V_{ub}^*V_{cs}^*\right) \; \approx \; -\frac{3}{2} x_{\rm u} x_{\rm d} \left(y_{\rm u} - y_{\rm d}\right)^2 \; \approx \; -3 x_{\rm u} x_{\rm d} |V_{cb}|^2 \, , \tag{29}
$$

and

$$
\begin{aligned}
\alpha &\equiv \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) \approx \arctan\left[\frac{3x_{\rm u}x_{\rm d}\left(y_{\rm u} - y_{\rm d}\right)^2}{2\left[x_{\rm u}^2 y_{\rm u}\left(y_{\rm d} - 3y_{\rm u}\right) + x_{\rm d}^2 y_{\rm d}\left(y_{\rm u} - 3y_{\rm d}\right)\right]}\right] \, , \\
\beta &\equiv \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) \approx \arctan\left[\frac{3x_{\rm u}x_{\rm d}\left(y_{\rm u} - y_{\rm d}\right)}{2x_{\rm u}^2 y_{\rm u} - x_{\rm d}^2\left(y_{\rm u} - 3y_{\rm d}\right)}\right] \, , \\
\gamma &\equiv \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) \approx \arctan\left[\frac{3x_{\rm u}x_{\rm d}\left(y_{\rm u} - y_{\rm d}\right)}{x_{\rm u}^2\left(y_{\rm d} - 3y_{\rm u}\right) - 2x_{\rm d}^2 y_{\rm d}}\right] \, .
\end{aligned}
\tag{30}
$$

Current experimental data yield $|V_{cb}| \approx 0.0415$, and the numerical exercises done above allow $R \equiv y_{\rm u}/y_{\rm d} \approx 0.3$, $x_{\rm u}^2 \approx 4.6 \times 10^{-2}$ and $x_{\rm d}^2 \approx 8.0 \times 10^{-4}$. Assuming $x_{\rm u}$ and $x_{\rm d}$ to have the opposite sign, we immediately obtain $\mathcal{J} \approx 3.1 \times 10^{-5}$. This result is in good agreement with the value of $\mathcal{J}$ given in Ref. [5]. In addition, we estimate the values of α, β and γ in Eq. (30) and arrive at $\alpha \approx 80^\circ$, $\beta \approx 23^\circ$ and $\gamma \approx 77^\circ$, which are essentially compatible with current experimental data [5].

Of course, our results depend on the diagonal perturbation patterns of $\Delta M_{\rm u}$ and $\Delta M_{\rm d}$ taken in Eq. (19). We shall carry out a systematic analysis of different forms of ΔM_x (for $x = l, \nu, {\rm u}, {\rm d}$) elsewhere [21], in order to find out simpler and more realistic textures of fermion mass matrices motivated by the S_3 flavor symmetry.

4 We have extended the standard electroweak model by introducing one Higgs triplet and realizing the type-II seesaw mechanism to generate finite neutrino masses, and then applied the S_3 flavor symmetry to both lepton and quark sectors. The resulting mass matrices of charged leptons, neutrinos, up-type quarks and down-type quarks have a universal form $M_x = \xi_x I + \zeta_x D$ with I being the identity matrix and D being the democracy matrix (for $x = l, \nu, {\rm u}, {\rm d}$). The observed mass hierarchies of charged fermions can therefore be understood in the S_3 symmetry limit with $\xi_x \ll \zeta_x$ (for $x = l, {\rm u}, {\rm d}$). We have also argued that $\xi_\nu \gg \zeta_\nu$ is likely to hold, implying a nearly degenerate mass spectrum for three neutrinos. Such a picture generally allows us to interpret why the quark flavor mixing matrix is close to I and why the lepton flavor mixing matrix may contain two large mixing angles originating from the diagonalization of D, after proper perturbations to M_l, M_ν, $M_{\rm u}$ and $M_{\rm d}$ are taken into account. However, the I terms of M_l, $M_{\rm u}$ and $M_{\rm d}$ used to be ignored or dictated to be zero by means of the $S_{3{\rm L}} \times S_{3{\rm R}}$ symmetry. We have stressed that this term is important for two reasons: on the one hand, it is intrinsical in the S_3 symmetry limit and should not be forgotten; on the other hand, it can significantly modify the smallest lepton flavor mixing angle θ_{13} or the off-diagonal elements of the quark flavor mixing matrix. To illustrate, we have discussed a diagonal perturbation ansatz of M_x to explicitly break the S_3 symmetry and calculate the mass spectra and flavor mixing patterns of leptons and quarks. We find that this simple ansatz is phenomenologically acceptable in the lepton sector, but it is difficult to produce correct $|V_{cb}|$ and $|V_{ts}|$ in the quark sector.

We have pointed out that a straightforward way to modify our ansatz is to consider different patterns of the perturbation matrices ΔM_l, ΔM_ν, $\Delta M_{\rm u}$ and $\Delta M_{\rm d}$. This requires a systematic analysis of fermion mass matrices based on the S_3 flavor symmetry, because there are many possibilities and the main criterion to select the favorite one is to see that its phenomenological consequences can be consistent very well with current experimental data on lepton and quark masses and flavor mixing parameters. We are going to do such a comprehensive but tedious analysis elsewhere [21].

One may also consider to combine the S_3 flavor symmetry with the type-I seesaw mechanism, the type-(I+II) seesaw mechanism or other mechanisms of neutrino mass generation [14, 27]. Of course, it is more interesting to simultaneously understand the lepton and quark flavor structures by means of the S_3 symmetry and its proper breaking mechanism. Some enlightening ideas in this kind of exercises are expected to be very helpful in the study of other discrete flavor symmetries and their consequences or implications on fermion mass generation, flavor mixing and CP violation.

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